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Intra-day market activity

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Abstract

This paper presents a study of intra-day patterns of stock market activity and introduces duration based activity measures for single stocks and multiple assets. The proposed measures involve *weighted durations*, i.e. times necessary to sell (buy) a predetermined volume or value of stocks. As such, they capture dependencies between intra-trade durations, transaction volumes and prices, and can be interpreted as liquidity measures. This approach allows us to highlight the intra-day variations of liquidity, its costs and volatility, and to develop a liquidity based asset ordering. The extension to a multivariate analysis yields new insights into the dynamics of portfolio liquidity by revealing various aspects of asset substitution, including the effects of correlated trade intensities of portfolio components. Several examples are used to show that in practice, the proposed liquidity measures become efficient instruments for strategic block trading and optimal portfolio adjustments. The paper also contains an empirical study of asset activity on the Paris Bourse. We examine the liquidity dynamics throughout the day and reveal the existence of periodic patterns resulting from world-wide interactions of major stock markets. In the multivariate setup, we report evidence on common patterns and correlations of trade intensities of selected stocks.

JEL classification: C41; D4

Keywords: Liquidity; High frequency data; Duration models; Activity and co-activity measures

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1. Introduction

The role of time in stock markets is inherently related to traders' preference for immediacy of transactions, which is often considered to be the major determinant of market liquidity (Black, 1971). The speed of market activity is nowadays higher then ever before, due in part to the electronic order matching systems implemented on major stock exchanges, including the Paris Bourse. Nevertheless, the trading process still reflects the individual decisions of traders, and is hence susceptible to fluctuate according to their reactions and changes in the environment. This paper investigates the intra-daily patterns of trade dynamics for selected stocks exchanged in Paris, within a broader context of market liquidity variation.

Traditionally the microstructure literature associates fluctuations in liquidity with the heterogeneity of market participants, and especially their reactions to news or information revealed by the decisions of other market participants (Kyle, 1985; Richardson and Smith, 1993). For example, the theory relates the activity of uninformed traders to periodic increases in liquidity at the openings and closures of stock markets, and the presence of the so-called day-of-the-week effects (like, for example, the Monday effect) (see Admati and Pfleiderer, 1988; Foster and Viswanathan, 1990). This relationship is confirmed by numerous empirical studies, including Foster and Viswanathan (1993) or Gallant et al. (1992) among others. Other contributions to the literature emphasize the role of market makers in providing liquidity, and discuss the informational content of the bid-ask spread considered as their remuneration (Grossman and Miller, 1988). Additional determinants of liquidity include transaction costs, taxes or individual decisions of companies on lowering the prices of their stocks in order to enhance the liquidity of their shares (Copeland, 1979; Anshuman and Kalay, 1996).

The existence of various liquidity determinants points to the complexity of any reasonable measure of liquidity. Despite the large body of literature, one can hardly find a definition indicating how to measure liquidity. Conventionally, an asset is considered liquid if it can be traded quickly, in large quantities with little impact on the market price (Keynes, 1930; Black, 1971; Yamey, 1985; Glosten and Harris, 1988). Note that this statement emphasizes three liquidity factors, i.e. the volume, time and price which have to be jointly considered in any trading strategy. For example, an investor who placed a limit order that was not filled instantaneously faces uncertainty not only regarding the price, but also the split of volume and the time (s) at which the parts of the order's volume will trade. Hence, these three factors have to be *jointly* accommodated by any aggregate measures of liquidity.[1] In practice, the standard summary statistics used by

[1] This issue is particularly important for comparative studies on competing stock exchange markets (see, e.g. Jones et al., 1994).

market regulators for liquidity assessment include, for instance, the daily traded volume, the turnover rate which is related to the speed of trading (direct liquidity measures), the daily average spread, or some idiosyncratic volatilities (indirect liquidity measures).

In this paper, we undertake the fairly challenging task of providing aggregate measures of liquidity which accommodate the three aforementioned factors. Two approaches can potentially be followed. In the first one, we consider an investor concerned primarily with the immediate execution of his order. The investor has to compensate the market for the priority to be served by offering a higher price than the market in a buy transaction or by selling below the market price. The price increment above (below) the market price may be viewed as a liquidity cost. Therefore, the instantaneous liquidity cost is measured here in terms of market price variation. In the second approach, we focus on the speed of market activity and examine the time required to trade a fixed quantity of shares or a predetermined value of offered shares. In this case, we measure the liquidity cost in terms of time necessary to execute an order of a predetermined size (value).

The first approach may be implemented by analyzing the order book. At each time t, orders from both sides of the market are ordered in queues. The absorptive capacities of the queues on the demand and supply sides arise as natural liquidity measures. Any trades involving volumes exceeding the capacity of the first line of the queues (usually called the depth of the market) create price effects. As an illustration, we show in Appendix C, the bid and ask curves corresponding to the order queues for an individual stock observed at 10:01 a.m., Tuesday, May 14th 1996 (Gouriéroux et al., 1998). The ask (resp. bid) curve can be interpreted as the unit price of a share, as a function of volume, which has to be paid in a buy (resp. a sell) transaction. The forms of bid and ask curves, the vertical distance between them at the origin (the spread), their slopes, degrees of convexity or asymptotes contain information about the immediate liquidity. The instantaneous approach fails however to provide insights into the market dynamics. For instance a large trade may induce a large instantaneous modification of price, but thereafter the price may bounce back quickly to the latent market price.

Our second approach involves the time required by the market to trade a fixed volume (or value) of shares. This concept is an analogue of the velocity of money introduced by Fisher and Marshall in the twenties, and further used in the quantitative theory of money (Friedman, 1959). We introduce duration-based activity measures to assess the time necessary to sell (buy) a predetermined volume or value of stocks. These measures capture dependencies between intra-trade durations, transaction volumes and prices, and can be interpreted as measures of liquidity. Our approach highlights intra-day patterns of liquidity costs and volatility, and provides a liquidity based asset ordering. The extension to a multivariate analysis yields additional insights into the dynamics of portfolio allocations. The analysis may involve all trades or be restricted to the demand or supply initiated trades, or to some specific types of orders (see also Biais et al. (1995) and Bisière and Kamionka (1997)).

Our empirical study of market activity is focused on trades of the Alcatel stock on the Paris Bourse in January 1995 and supplemented by data on Saint-Gobain stock for the multivariate framework. We show how trade intensity fluctuates throughout the day, revealing periodic patterns resulting from world-wide interactions of major stock markets. In the multivariate setup, we also study common patterns in trade intensities of several stocks.

Our paper is organized as follows. In Section 2, we develop a periodic model of trading activity and use duration data and trading rates to build simple measures of stock activity. We apply this approach to illustrate intra-day periodic patterns on the Paris Bourse and report results regarding intra-day sub-periods characterized by the absence of relevant information. In Section 3, we extend the notion of duration and introduce volume and value-weighted durations. This section discusses new liquidity measures, the ordering of assets with respect to their trading activity, and estimators of liquidity costs and duration volatility. Section 4 covers the multivariate setup and defines the co-activity measures for multiple assets. It provides a decomposition formula for identification of the marginal and cross effects of trading frequencies and volumes. Section 5 concludes the paper. Technical details appear in Appendices.

2. Time based activity measures

This section examines asset activity determined by the rate of trade arrivals. We present a periodic model of trading activity below and introduce basic activity measures for a single asset (stock). These measures involve alternatively the intra-day trade count data, or series of durations defined as times between successive transactions. In practice, both types of measures produce equivalent outcomes. It has to be emphasized that while the trade count based measures are computationally less intense, the duration based measures lend themselves to further generalizations introduced in Section 3.

As an illustration of our method, we present an empirical study of intra-day patterns of Alcatel's trading activity. Our sample covers a period of 18 days in January 1995 during which the stock market was closed for two weekends (January 7th and 8th, and 14th and 15th). Therefore, we consider trades recorded over fourteen working days, denoted by m, where $m = 1, \dots, 14$. Their distribution remains almost invariant throughout the days of the week, i.e. Monday through Friday, which allows us to disregard the day-of-the-week effect (Admati and Pfleiderer, 1989; Yadav and Pope, 1992; Ghysels et al., 1997a).

The intra-day-clock time on day m is denoted by t, and is reset to zero after each market closure. It measures the part of the day during which the market operates (i.e., from 10:00 a.m. to 5:00 p.m., equivalent to 25 200 s). On every trading day we observe, after eliminating the opening transactions, a sequence of trades which number is N_m on day m. The individual trades are indexed by $n, n = 1, \ldots, N_m$, and the associated trading times are denoted by $d_n(m)$. The duration between the successive ticks $n-1$ and n is

$$\tau_n(m) = d_n(m) - d_{n-1}(m).$$

We also introduce the intra-day counting process, $N_t(m)$ indicating the number of transactions concluded by time t on day m.

2.1. Duration models

The intra-trade durations are determined by the trade intensity. It is assumed to be a deterministic continuous function of time $\lambda(t)$, where $t \in [0, 25\,200]$ is restricted to span one trading day. In our approach, λ does not depend on the history of the trading process, represented, for example, by lagged durations, lagged returns, lagged traded volumes, or latent stochastic factors (see Engle and Russell (1997,1998) or Ghysels et al. (1997c) for dynamic duration models). We also disregard the day of the week effect, which is negligible due to the choice of a suitable number of successive days. The trade intensity obeys the following assumption on the frequency of trades:

Assumption 1: The trade arrivals in the infinitesimal time interval $[t, t + \mathrm{d}t]$ are such that

$$\Pr\begin{bmatrix}\text{one trade between} \\ t \text{ and } t + \mathrm{d}t, \text{ day } m\end{bmatrix} = \lambda(t)\mathrm{d}t + \mathrm{o}(\mathrm{d}t), \tag{2.1}$$

and

$$\Pr\begin{bmatrix}\text{strictly more than one trade between} \\ t \text{ and } t + \mathrm{d}t, \text{ day } m\end{bmatrix} = \mathrm{o}(\mathrm{d}t), \tag{2.2}$$

where $\mathrm{o}(\mathrm{d}t)$ is negligible with respect to $\mathrm{d}t$.

Under Assumption 1, the cumulated arrival rate is

$$\Lambda(t) = \int_0^t \lambda(u)\,\mathrm{d}u.$$

The cumulated arrival rate reveals the past intensity of trades. The impact of past trades on the size of current intra-trade durations is defined by the survivor function of $\tau_n(m)$ conditional on $\tau_{n-1}(m), \dots, \tau_1(m)$ (Andersen et al., 1992):

$$\begin{aligned}\Pr[\tau_n(m) \geqslant \tau \mid \tau_{n-1}(m), \dots, \tau_1(m)] &= \exp\left[-\int_{d_{n-1}(m)}^{d_{n-1}(m)+\tau} \lambda(u)\,\mathrm{d}u\right] \\ &= \exp - \{\Lambda[d_{n-1}(m)+\tau] - \Lambda[d_{n-1}(m)\} \\ &= S(\tau \mid d_{n-1}(m)) \quad \text{(say)} \end{aligned} \tag{2.3}$$

The last formula implicitly involves past cumulated durations since $d_{n-1}(m) = \sum_{p=1}^{n-1} \tau_p(m)$. The presence of this term can be interpreted as a random walk feature in the duration dynamics.

The empirical analogues of the trade intensity λ and the survivor function $S(\tau, d)$ are proposed as elementary measures of the trading activity. If we assume that one unit of volume is exchanged in each transaction, the intensity λ becomes essentially the trade velocity (see e.g. Garman (1976), Mendelson (1982), Glosten and Milgrom (1985), Easley and O'Hara (1992) and Foucault (1996) for such a restrictive assumption). It is a scalar measure varying throughout the day. On the other hand, the empirical survivor function evaluated at various intraday times d arises as a functional activity measure. Intuitively, when the trading activity decreases, the market slows down and the survivor function takes higher values. We present below consistent estimators of these two measures.

2.2. Estimation methods

The trade intensity and survivor function introduced above, can be consistently estimated using nonparametric methods. The estimators in our approach are kernel smoothed to eliminate local discontinuities. We call kernel a positive bounded function, such that $\int K(u)\,\mathrm{d}u = 1$. A typical example is the Gaussian kernel:

$$K(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right).$$

The kernel function enhances the graphical illustration of market dynamics. It helps to attenuate occasional irregularities and reveals underlying intra-day patterns. We introduce first the kernel based trade intensity estimator followed by the kernel based survivor function estimator.

2.2.1. Trade intensity estimator

The estimator of the trade intensity rate $\lambda(t)$ at any time t of the trading day is derived by smoothing the average numbers of trades recorded at that time over several days. In this approach, data used for estimation consist of intra-day trade counts. We know that the cumulated rate $\Lambda(\cdot)$ is equal to the unconditional

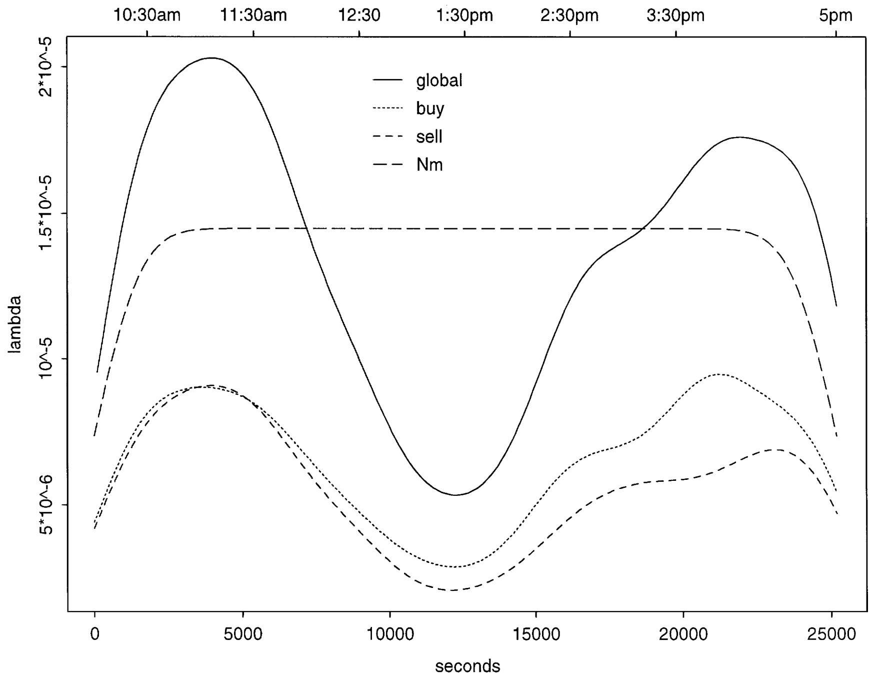

Fig. 1. Estimated trading rates $\hat{\lambda}(t)$, Alcatel.

expectation of the counting process $N_t(m)$, $t \in [0, 25\,200]$. Therefore, an average of trade arrivals computed over M days yields a consistent estimator of $\Lambda(\cdot)$, when M tends to infinity:

$$\hat{\Lambda}(t) = \frac{1}{M} \sum_{m=1}^{M} N_t(m), \quad t \in [0, 25\,200]. \tag{2.4}$$

Since $\hat{\Lambda}(\cdot)$ is not differentiable, the derivation of a consistent estimator of $\lambda(\cdot)$ requires preliminary smoothing. A consistent kernel-based estimator of $\lambda(\cdot)$ is

$$\hat{\lambda}(t) = \frac{1}{M} \sum_{m=1}^{M} \sum_{n=1}^{Nm} \frac{1}{h} K\left[\frac{d_{n-1}(m) - t}{h}\right], \tag{2.5}$$

where h is the bandwidth assumed to tend to zero at a suitable rate when M tends to infinity. In this formula, the closer the observed trading dates are to date t, the higher become the associated weights.

As an illustration, Fig. 1 displays the trading intensity for the Alcatel stock estimated using a Gaussian kernel with a bandwidth $h = 22$ min.[2] The "global

[2] Note that while the smoothness of the trading rate is altered when considering a smaller bandwidth, the intra-day pattern remains the same.

rate" corresponds to the intensity measure based on all trades, whereas those indicated on the plot as "buy" and "sell" distinguish between the buyer and seller initiated trades. We see that trade intensities are not constant and feature strong intra-day seasonal patterns.[3] They also seem to be almost parallel to each other suggesting the presence of similar intra-day patterns.

Finally, for comparison and a quick assessment of the kernel effect, we illustrate a simulated trading scheme where $Nm \sim 600$ ticks are equally spaced during the day. The dashed line (curve "Nm") shows the estimated intensity of this trading scheme. We observe a straight line over the central part of the figure, spuriously bent at the ends due to the kernel effect, roughly during the first and last 30 min of trades.

The market microstructure theory does not provide a straightforward explanation of periodic fluctuations of intra-day activity. The literature relates the concentration of trades on some markets (Pagano, 1989) and during some subperiods (Admati and Pfleiderer, 1988) to the behavior of traders classified as informed or uninformed. Also, the differentiation of transaction costs is used as another argument in the theoretical work. In general, the theory fails to explain why the sequence of peaks and troughs in the market activity appears repeatedly on different days. This emphasizes the role of empirical studies of trade dynamics in revealing the market regularities. Our results indicate that the most active periods on the Paris market correspond to the opening of the London Stock Exchange around 10:30 a.m. and the New York Stock Exchange around 3:30 p.m. The least active one is the lunch break. It may be noted that we find an M-shaped curve instead of the commonly reported U-shaped curve [see, for example Stephan and Whaley (1990), Wei (1992), Foster and Viswanathan (1993) and Engle and Russell (1997,1998)). Indeed the Paris market accelerates after the opening and slows down 10 min before the closure (see Fig. 2). This last effect can not be observed directly in Fig. 1 due to smoothing.

In our data set we only retain trades recorded after the opening. It is necessary to do so because of the different order matching procedures: the opening price at 10:00 a.m. comes from a call auction whereas after the opening, the market switches to a continuous matching procedure. If the opening trades were included, the shape of the curve would be different although a perfect U-shape would not be reproduced (see Fig. 3). This is likely due to the fact that we consider one single stock. Indeed, when we consider aggregate trades on multiple assets, the M-shape turns into the U-shape; this latter form may be considered a result of an aggregation bias.

Finally, we observe in Fig. 1 a small local peak around 2:30 p.m. It likely indicates an activity increase due to the arrival of new market participants before the opening of the NYSE. This pattern can be compared to Fig. 4 which

[3] The seasonalities can also be detected in frequencies of order arrivals [see Appendix A].

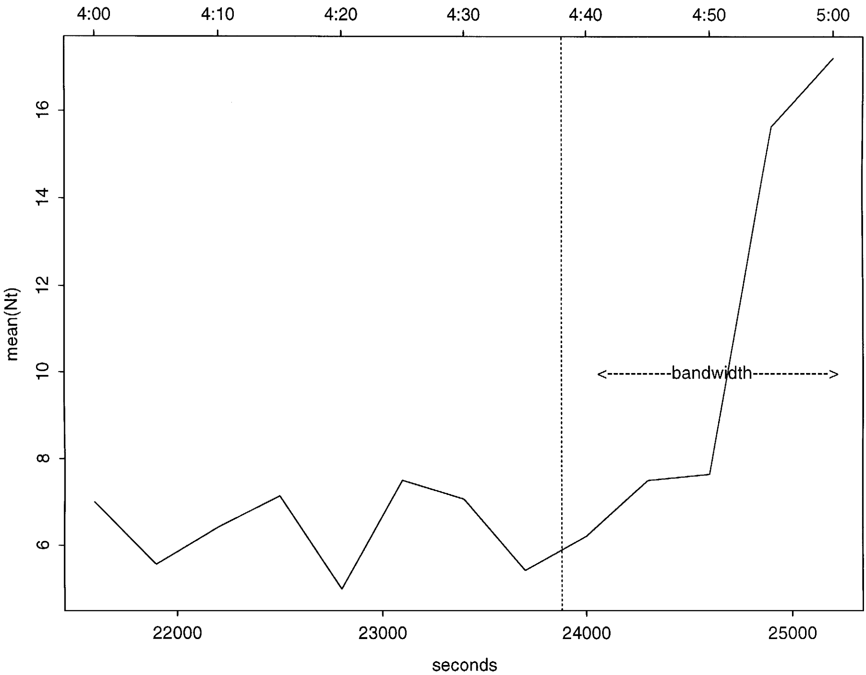

Fig. 2. Average number of transactions over 5 min intervals before the market closure, over 14 trading days, January 1995, Alcatel.

shows the skewness of the return distribution during the day. This distribution is generally symmetric except for this moment when a large right asymmetry is observed. This asymmetry may be due to a mixture of distributions at this time indicating the presence of heterogeneous behaviors [see Fig. 4].

2.2.2. Survivor function estimator

We now turn our attention to the estimation and measurement of activity using the conditional survivor function $S(\cdot|d)$. This approach requires data on intra-day trading times. Recall that $S(\tau|d)$ is the probability of waiting at least τ for the arrival of the next trade conditional on the date $d_{n-1}(m)$ of the last transaction. It may be approximated by

$$\hat{S}(\tau|d) = \frac{1}{M}\sum_{m=1}^{M} \frac{\sum_{m=1}^{N_m} \mathbb{1}_{\tau_n(m)>\tau} \mathbb{1}_{d-h<d_{n-1}(m)<d+h}}{\sum_{n=1}^{N_m} \mathbb{1}_{d-h<d_{n-1}(m)<d+h}}, \tag{2.6}$$

where $\mathbb{1}$ denotes an indicator function. However, again due to discontinuities which result from averaging, it is preferable to replace the indicator function by

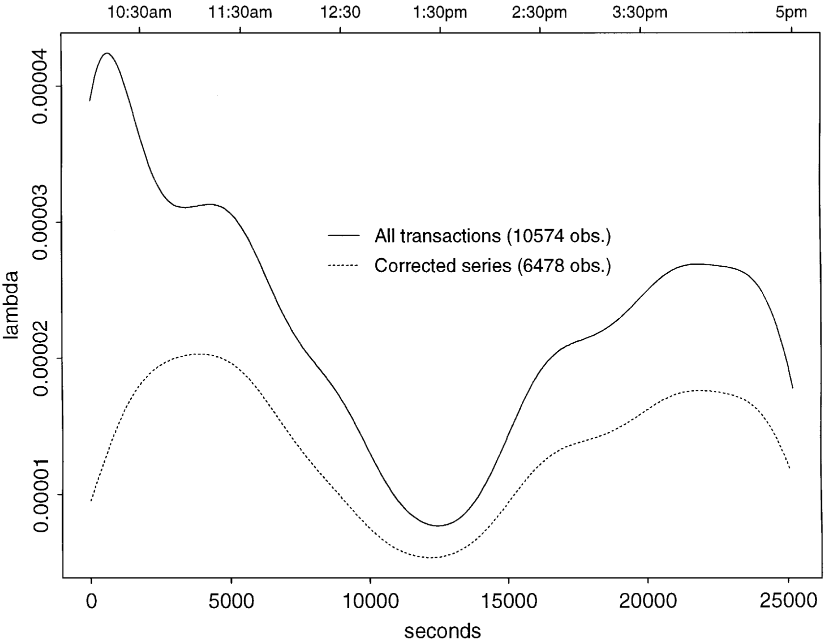

Fig. 3. Estimated trading rate $\hat{\lambda}(t)$ when the opening trades are included and the "large" trades are not aggregated, Alcatel.

a kernel function (Silverman, 1986):

$$\hat{S}(\tau|d) = \frac{1}{M}\sum_{m=1}^{M} \frac{\sum_{n=1}^{N_m} \mathbb{1}_{\tau_n(m)>\tau} K\left(\frac{d_{n-1}(m)-d}{h_1}\right)}{\sum_{n=1}^{N_m} K\left(\frac{d_{n-1}(m)-d}{h_1}\right)}. \tag{2.7}$$

By applying a second smoothing, the conditional p.d.f. of the duration can easily be derived. The kernel density estimator is

$$\hat{f}(\tau|d) = \frac{1}{M}\sum_{m=1}^{M} \frac{1}{h_2} \frac{\sum_{n=1}^{N_m} K\left(\frac{\tau_n(m)-\tau}{h_2}\right) K\left(\frac{d_{n-1}(m)-d}{h_1}\right)}{\sum_{n=1}^{N_m} K\left(\frac{d_{n-1}(m)-d}{h_1}\right)}. \tag{2.8}$$

The estimates of the survivor or density functions provide information on the behavior of intra-trade durations throughout the trading day and, as such, measure the market speed. Figs. 5 and 6 display the empirical density functions at various times estimated using a Gaussian kernel and two bandwidths fixed at

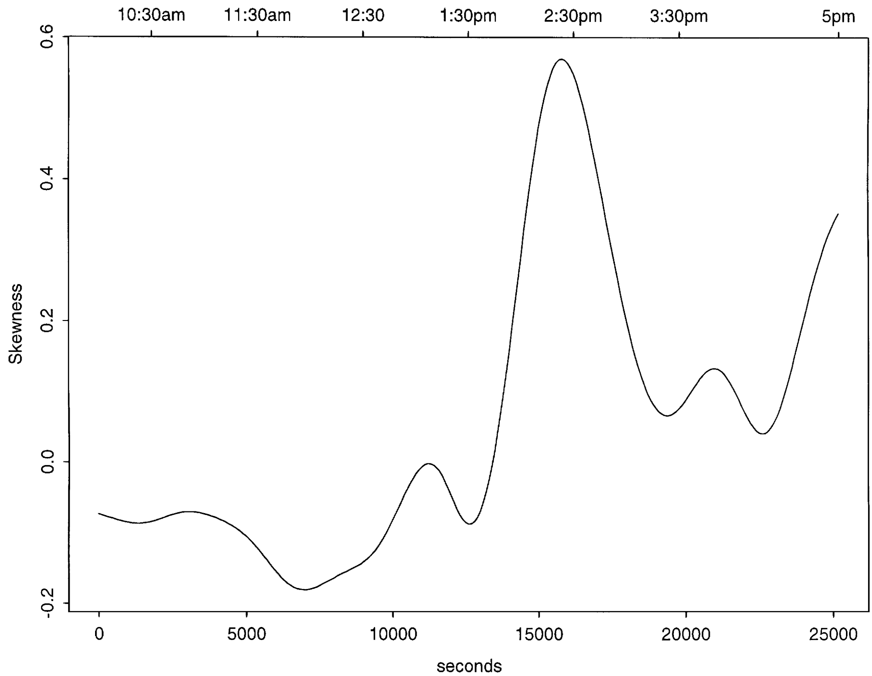

Fig. 4. Intra-day skewness, over the period ranging from January the 3rd to the 19th 1995, Alcatel.

$h_1 = 22$ min and $h_2 = 43$ s. Inference based on the survivor function and the trade intensity, discussed in the last subsection yield similar results. Let us focus now on the activity decrease during the lunch break. The market slowdown is reffected by long upper tails of duration distributions during this period of the day. This indicates more frequent occurrence of very large durations between trades recorded during the lunch break.

2.3. Non-informative subperiods

The estimators of the survivor function introduced above may be used for an assessment of the informational content of data recorded over a fixed interval. For example, let us check if the news arriving during a given subperiod $[d, d+\tau]$ contains additional information for predicting the future trading activity. We claim that this subperiod is non-informative if the forecasts evaluated at d and at $d+\tau$ of future activity at $d+\tau$ and later coincide. This condition may be written in terms of the survivor functions:

$$S(\tau|d+\delta) = \frac{S(\tau+\delta|d)}{S(\delta|d)} \quad \forall \tau > 0. \tag{2.9}$$

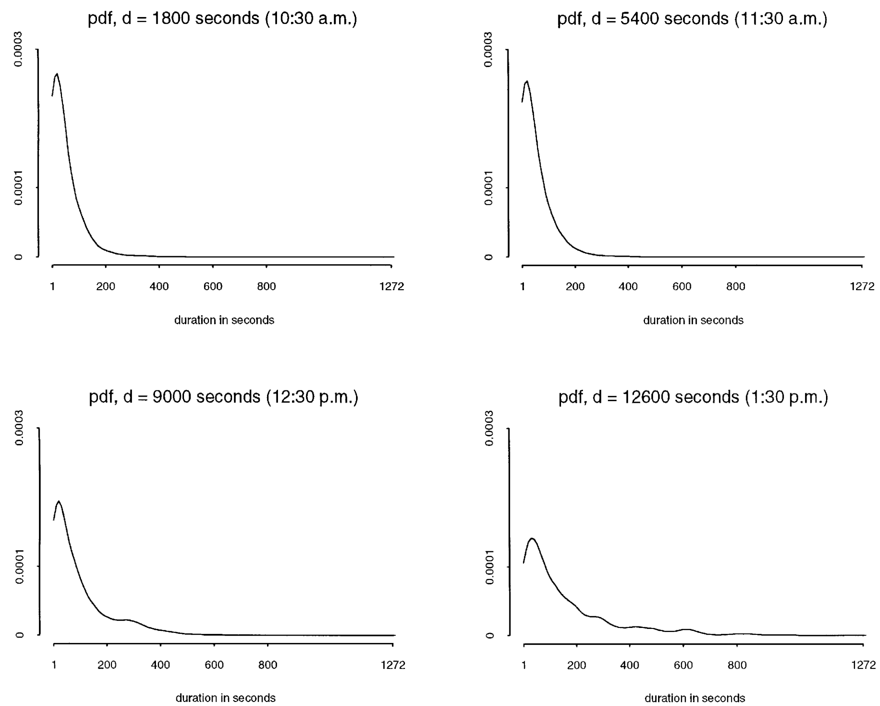

Fig. 5. Estimated conditional p.d.f. of the durations for $d = $ 10:30 a.m., 11:30 a.m., 12:30 p.m. and 1:30 p.m., Alcatel.

If the right and left sides of (2.9) are equal, duration data feature a lack of memory. This term, borrowed from the duration literature, describes the absence of temporal dependence.[4] We show in Fig. 7 the estimated functions $\hat{S}(\cdot|d+\delta)$ and $\hat{S}(\cdot+\delta|d)/\hat{S}(\delta|d)$ for $d =$ 1:00 p.m. and $d+\delta =$ 1:15 p.m. The rationale for this particular choice is to verify if the lunch sub-period is a non-informative sub-period of the day. The Figure indicates that the equality of predictions does not hold and that the trader's forecast made at 1:00 p.m. overestimates the required time to trade, or equivalently underestimates the future market activity.

3. Weighted activity measures

In the previous section, the analysis of market activity was restricted to the trade arrival process and did not account for any other trade characteristics. We

[4] Exponentially distributed durations satisfy this condition for any d, δ, τ.

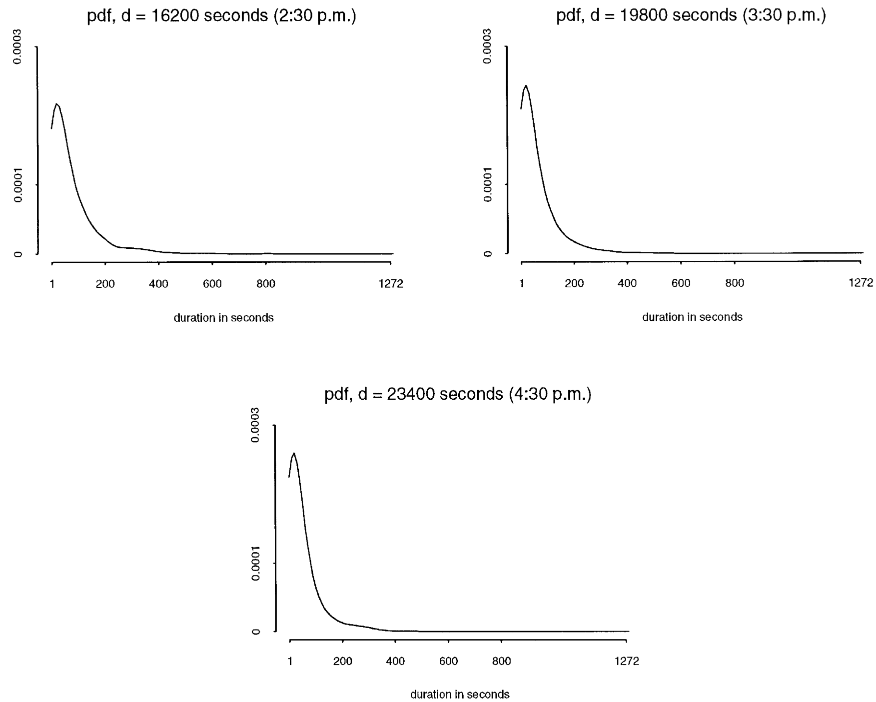

Fig. 6. Estimated conditional p.d.f. of the durations for $d = 2{:}30$ p.m., 3:30 p.m. and 4:30 p.m., Alcatel.

now address this issue and redefine the duration based activity measures introduced in Section 2.2. The approach is straightforward in the sense that the concept of activity measures remains unchanged. The crucial refinement consists in adapting the definition of durations to capture the volume and price effects. This approach is motivated by dependencies between the intra-trade durations and trade prices and volumes perceived by individual traders.

Consider an investor who places a sell limit order at price $p^*_{t_0}$, at time t_0, for one share (a unit volume). If this order is not immediately executed while remaining on top of the order book, the trader is interested in finding out the waiting time to trade. This time is $\tau(t_0, p^*_{t_0}) = \inf\{\tau\colon p_{t_0+\tau} \geqslant p_{t_0^*}\}$, and depends on the future price evolution. If the order involves, instead of one share, a larger quantity of shares $v > 1$, the waiting time will also be determined by the future behavior of the traded volume process. Hence, to accommodate jointly both effects, we need to evaluate the time to trade, weighted appropriately by the price and volume.

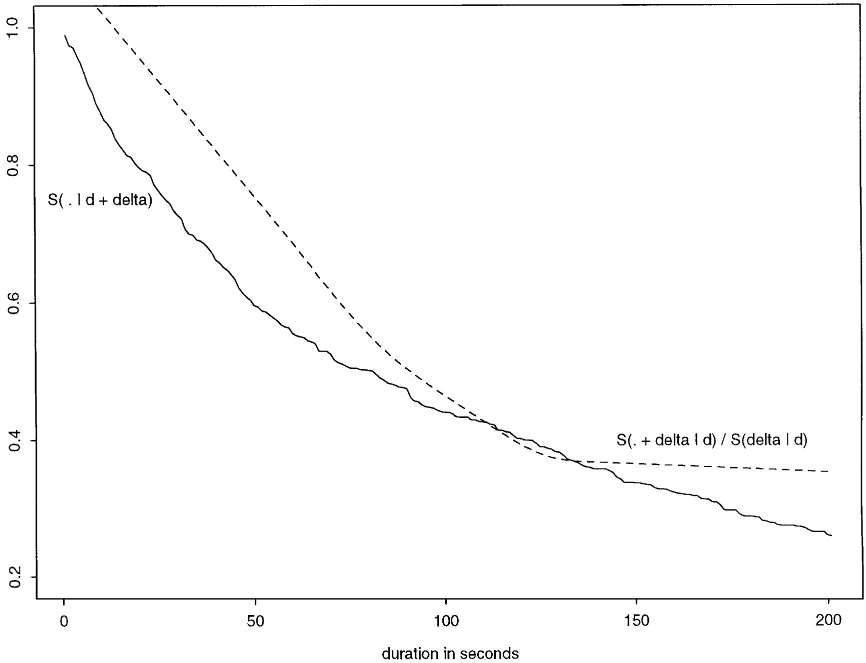

Fig. 7. Comparison of predictions at $d = $ 1:00 p.m. and $d + \tau = $ 1:15 p.m., Alcatel.

In this section, we introduce *weighted durations* which represent the time required by the market to trade a fixed volume v or a fixed capital w, and propose volume and value weighted market activity measures. The volumes are numbers of shares traded while the trade values are numbers of shares multiplied by asset prices. These measures can easily be redefined per volume or value unit for comparison of liquidity at different times of the day, across assets (using the common monetary denominator) or across markets. These market weighted activity measures are likely related to the individual weighted waiting times, but the analysis of this complicated aggregation problem is clearly out of the scope of this paper.

3.1. Definition of the measures

Let $v_n(m)$ denote the volume traded at time $d_n(m)$. By summing up volumes of individual trades for a total count of $N_t(m)$ transactions, we obtain the cumulated traded volume:

$$V_t(m) = \sum_{n=1}^{N_t(m)} v_n(m), \tag{3.1}$$

i.e. the volume exchanged on day m by t. Similarly, we define the cumulated capital:

$$W_t(m) = \sum_{n=1}^{N_t(m)} p_n(m) v_n(m), \tag{3.2}$$

where $p_n(m)$ is the price of transaction n, day m.

The extended durations defined below represent the time required by the market to trade a fixed volume v in shares or w in capital. They also depend on the time t at which the order is placed and hence vary throughout the day. We define the volume duration:

$$\tau_{vol}(t, v) = \inf\{\tau\colon V_{t+\tau}(m) \geqslant V_t(m) + v\}, \tag{3.3}$$

as the time necessary to observe an increment v of cumulated volume. The capital duration is

$$\tau_W(t, w) = \inf\{\tau\colon W_{t+\tau}(m) \geqslant W_t(m) + w\}, \tag{3.4}$$

the time necessary to observe an increment w of cumulated capital.[5]

The empirical densities of volume or capital durations arise as new measures of trade activity incorporating the price and volume effects. The p.d.f. of volume-weighted duration can be estimated consistently for any volume v via

$$\hat{f}(\tau|t) = \frac{1}{M} \sum_{m=1}^{M} \frac{1}{h_2} \frac{\sum_{n=1}^{N_m} K\left(\frac{\tau_{vol}(d_{n-1}(m), v) - \tau}{h_2}\right) K\left(\frac{d_{n-1}(m) - t}{h_1}\right)}{\sum_{n=1}^{N_m} K\left(\frac{d_{n-1}(m) - t}{h_1}\right)}. \tag{3.5}$$

The volume duration density is useful to evaluate and compare chances to trade a predetermined volume v within a fixed interval τ at different times of the day. This approach is illustrated in Figs. 8 and 9 . They display the time varying empirical densities of weighted durations evaluated for a volume of 10 500 shares of the Alcatel stock (corresponding to the average volume exchanged within 15 min of trades).

[5] The introduction of the cumulated volume and capital and of the corresponding weighted durations is related to the idea of market time deformation (see, e.g. Mandelbrot and Taylor, 1967; Clark, 1973; Stock, 1988; Daracogna et al., 1994; Ghysels et al., 1996,1997a,b,1998; Müller et al., 1994,1997). Indeed the processes $N_t(m)$, $V_t(m)$, $W_t(m)$, with t varying, are non-decreasing processes and may be used as stochastic changes of time. The associated durations, such as τ_{vol}, are simply the inverse time scales. Indeed we have

$$\tau_{vol}(0, v) = Inf\{\tau\colon V_\tau > v\}, \text{ or more generally } \tau_{vol}(t, v) = \tau_{vol}(0, V_t(m) + v) - t.$$

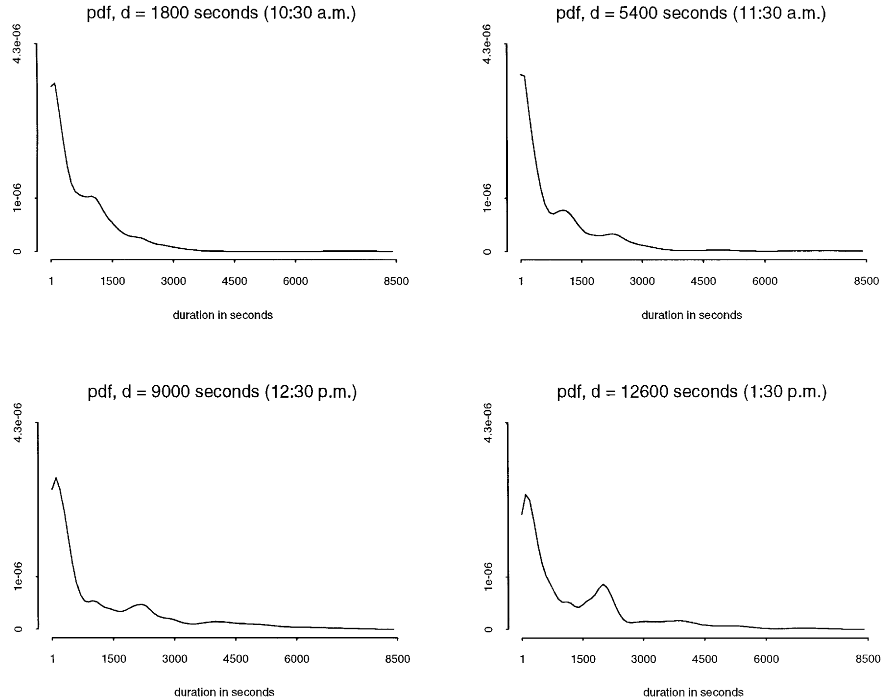

Fig. 8. Estimated conditional p.d.f. of volume weighted durations for $v = 10\,500$ shares and $d =$ 10:30 a.m., 11:30 a.m., 12:30 p.m. and 1:30 p.m., Alcatel.

While the conditional densities of the simple durations at 10:30 a.m. and 4:30 p.m. are about the same (see Figs. 5 and 6), this is not the case for the conditional p.d.f. of the volume-weighted durations. From the work of Biais et al. (1995), we know that large trades occur more frequently in the afternoon, whereas small trades are more frequent in the morning. The durations between trades defined in Section 2 do not depend on the trade size, and hence the resemblance of their densities at 10:30 a.m. and 4:30 p.m. is due to the fact that the decrease in small trade frequency in the afternoon is compensated by the increase of the large trade frequency and conversely in the morning. Moreover, Fig. 8 shows that there is much less intra-day variation in the conditional p.d.f.'s of the volume-weighted durations. Indeed when the overall number of trades decreases between 1:00 p.m. and 2:00 p.m., this decrease is caused by a low number of small orders rather than a declining quantity of large orders. Biais et al. (1995) propose four explanations of the different trading frequencies of small and large trades within the day: (1) the discovery of the daily fundamental value, which leads to trade large volumes as uncertainty decreases, (2) the evaluation of fund managers' performance with respect to the closing price,

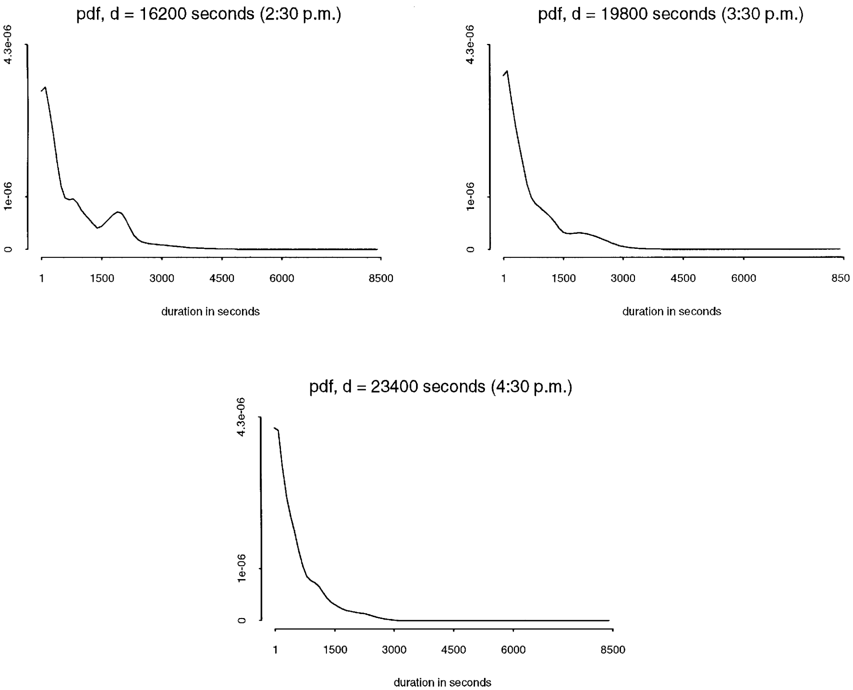

Fig. 9. Estimated conditional p.d.f. of volume weighted durations for $v = 10\,500$ shares, and $d =$ 2:30 p.m., 3:30 p.m. and 4:30 p.m., Alcatel.

(3) bank transmissions of small orders from their customers at the beginning of the day, and (4) the activity of strategic investors splitting their orders differently during the day.

3.2. Market liquidity costs

The weighted durations yield measures of liquidity costs and measures of volatility of the times to trade. The liquidity cost per unit of volume or value can be approximated by the average weighted duration corresponding to that volume or value. Similarly, the dispersion of weighted durations provides a measure of volatility of waiting times for trades of a fixed size or value.

The liquidity costs are defined as the ratio of the expected weighted duration and the volume v or the capital w:

$$m_{vol}(t, v) = \frac{E[\tau_{vol}(t, v)]}{v}, \quad m_W(t, w) = \frac{E[\tau_W(t, w)]}{w}. \tag{3.6}$$

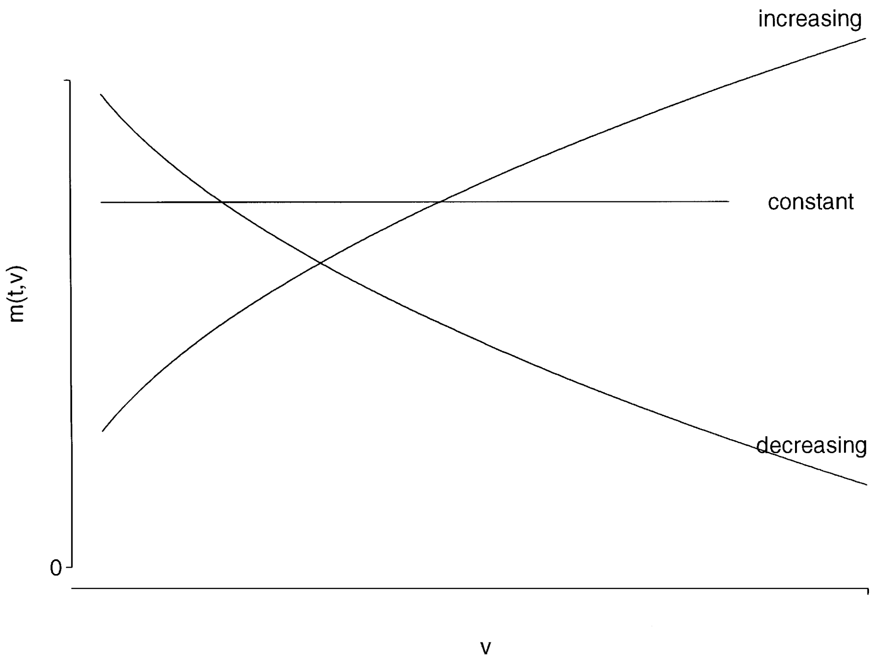

Fig. 10. Liquidity cost.

They can easily be estimated by averaging durations corresponding to various volumes (capitals). As such, they represent average times of execution of orders of a given size. We know that assets may feature constant, decreasing or increasing market liquidity costs as illustrated in Fig. 10.

The level, slope, and convexity of liquidity cost functions, may vary depending on the asset and the time of day. In the first column of Fig. 11, we plot the average durations as a function of volume for three different times of the day, distinguishing the type of order that initiates the trade. The figure reveals the decreasing liquidity costs of the Alcatel stock. In general, large trades are less costly in terms of the order execution time than small trades. We also observe that liquidity costs vary throughout the day. They are high in the morning and seem to diminish in time, so that late afternoon trades become the least expensive. This observation is valid for trades of any size. This result is consistent with the increase in the frequency of large trades in the afternoon. According to the microstructure literature, the uninformed traders exploit the high level of competition between the informed traders late in the day.

Investors may also be concerned with the volatility of times to trade. It represents the risk regarding the order execution time, comparable to the risk on an asset's return. Intuitively, traders have preferences for short expected trading

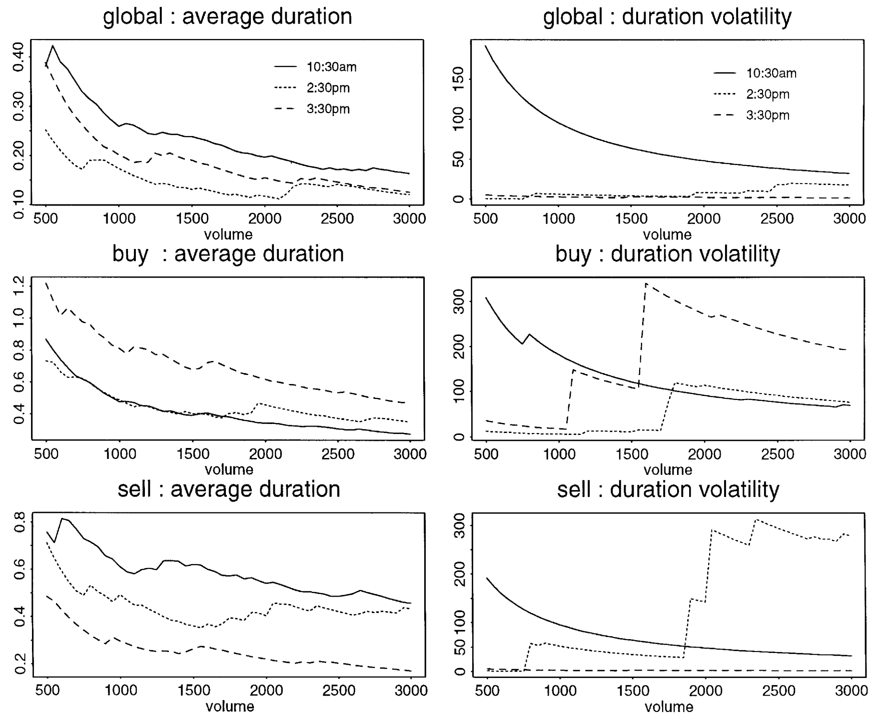

Fig. 11. Evolution of the estimated average durations and duration variability for t = 10:30 a.m., 2:30 p.m. and 3:30 p.m.

times and small duration variabilities. The concept of volume (value) weighted durations allows us to identify risks related to various order sizes. We call volume/value duration volatilities the ratios of their variances and squared volumes (values):

$$v^2_{vol}(t, v) = \frac{\mathsf{V}\,[\tau_{vol}(t, v)]}{v^2}, \quad v^2_W(t, w) = \frac{\mathsf{V}\,[\tau_W(t, w)]}{w^2}. \tag{3.7}$$

The volume duration volatilities can be estimated by computing average duration variances for different volume levels. The second column of Fig. 11 shows the intra-day variation of volume duration volatility for Alcatel. At the aggregate level the volatility is much higher in the morning than during the rest of the day for almost all volume levels, except for very large trades. The buyers face much more duration volatility in the evening compared to the sellers especially when purchasing large quantities of shares. The volatility at 2:30 p.m. increases strongly for large trades due to the anticipated opening of the NYSE. This effect seems to affect more the sellers than the buyers. The volatility curves for the afternoon trades are stepwise functions indicating traders' preferences for selected blocks of volume, consisting of less than 1000 shares, 1000–1500 shares and more than 1500 shares.

3.3. Asset ordering

The optimal portfolio allocation requires instruments allowing a trader to assess the liquidity of an asset in time and compare liquidities of various assets. The ordering defined below relies on the weighted durations and hence accommodates the joint effects of time, volume and value. It can be interpreted as an analogue of asset classification with respect to the risk on returns.

Definition 3.1. The asset is more active at t_1 than at t_0 if and only if

$$\Pr[\tau_{vol}(t_1, v) > \tau] < \Pr[\tau_{vol}(t_0, v) > \tau], \quad \forall \tau, v \tag{3.8}$$

$$\Leftrightarrow S_{vol}(\tau; t_1, v) < S_{vol}(\tau; t_0, v), \quad \forall \tau, v, \tag{3.9}$$

where $S_{vol}(\tau; t, v)$ is the survivor function of $\tau_{vol}(t, v)$.

When two different assets are considered, times to trade per share are not comparable due to the potentially different values of the shares. Instead, monetary units provide a common denominator and various assets may hence be compared in terms of their value duration:

Definition 3.2. At time t, asset 1 is more active than asset 2, if and only if

$$\Pr[\tau_W^1(t, w) > \tau] < \Pr[\tau_W^2(t, w) > \tau], \quad \forall \tau, w, \tag{3.10}$$

$$\Leftrightarrow S_W^1(\tau; t, w) < S_W^2(\tau; t, w), \quad \forall \tau, w, \tag{3.11}$$

where $S_W^i(\tau; t, v)$ is the corresponding value survivor function.

The above orderings are closely related to the stochastic dominance of order 1. Indeed asset 1 is more active than asset 2 at time 1, if and only if $\tau_W^2(t, w)$ stochastically dominates $\tau_W^1(t, w)$ at order one, for any w (see Brunelle and Vickson (1975) for stochastic dominance, and Fourgeaud et al. (1987) for the application of these concepts in duration models). These orderings are not complete. For instance we may have:

$$S_W^1(\tau; t, w) \leqslant S_W^2(\tau; t, w), \quad \forall \tau, \forall w \leqslant w_0,$$

$$\text{and } S_W^1(\tau; t, w) \geqslant S_W^2(\tau; t, w), \quad \forall \tau, \forall w \geqslant w_0,$$

In this case, asset 1 is more liquid than asset 2 when small quantities are traded while we have a reverse ranking for large quantities. This situation can be

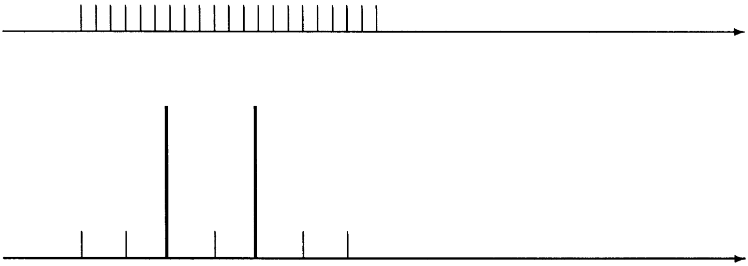

Fig. 12. Distinct trading patterns.

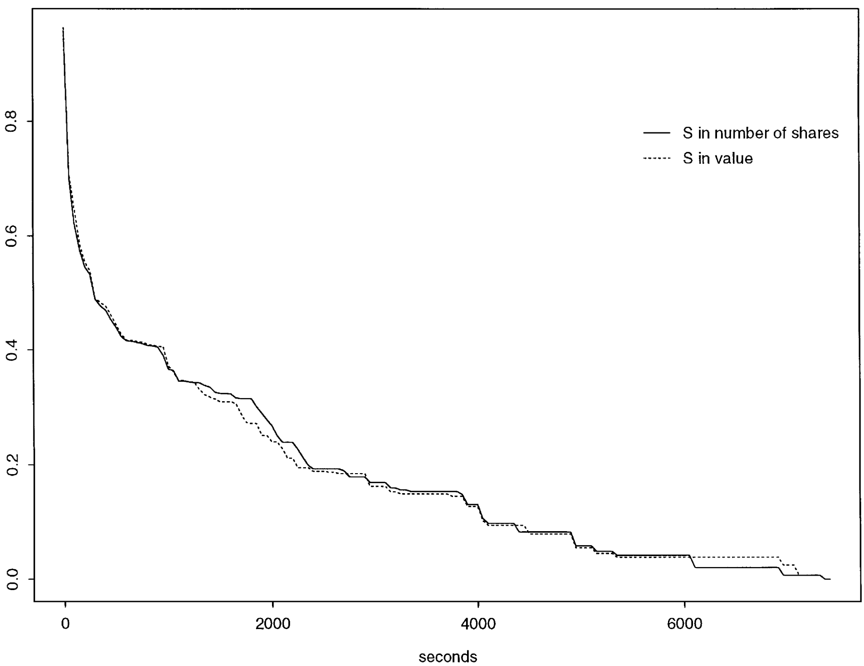

Fig. 13. Estimated volume and value survivor functions at 12:30 p.m., $v = 10\,500$ shares and $w = 4889\,400$ FF, Alcatel.

illustrated by two extreme trading patterns plotted above (Fig. 12). In the top panel, we observe regular frequent trades of small quantities while, in the lower panel, we observe less frequent trades, but periodically large quantities are exchanged. Hence, the first pattern is preferred for trading small quantities whereas the second one is better for large transactions.

In Fig. 13, we compare the empirical survivor functions $\hat{S}_{vol}$ and $\hat{S}_W$ estimated at fixed volume and capital levels. Both functions were evaluated at 12:30 p.m.

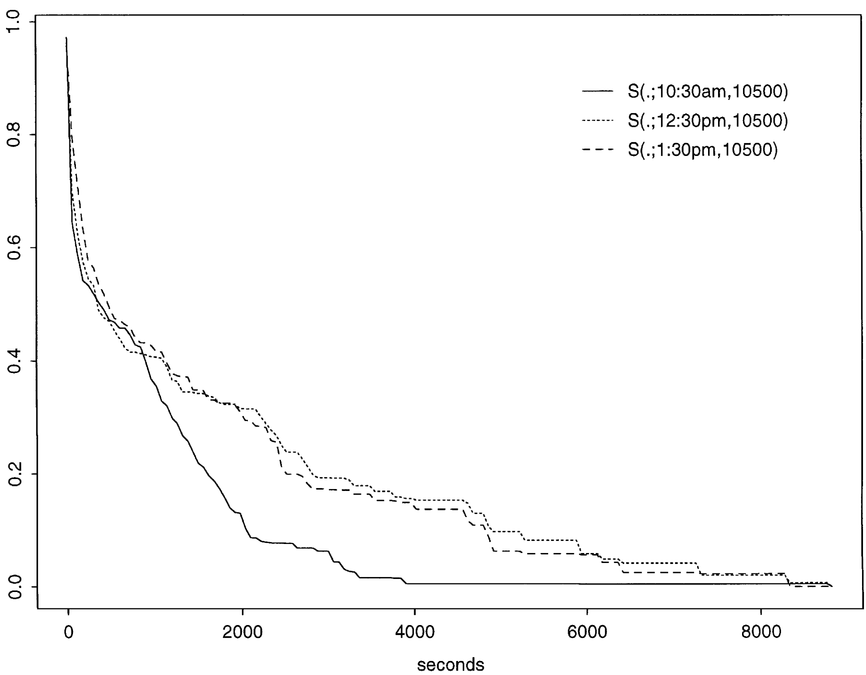

Fig. 14. Estimated survivor functions of volume duration for $v = 10\,500$, and t varying.

The volume v is again 10 500 shares while the function $\hat{S}_W$ was computed for the value of 4 889 400 FF, i.e. the average volume traded in 15 min multiplied by the average price of Alcatel over the entire sample period. The curves seem to overlap for most of the duration lengths. Occasionally we observe the appearance of little vertical divergences between the curves due to volume heterogeneity. This confirms that the volume and value survivor functions are complementary instruments for graphical analysis.

Fig. 14 illustrates the time-varying liquidity for Alcatel. We display the volume survivor functions evaluated at 10:30 a.m., 12:30 p.m., and 1:30 p.m. Alcatel appears to be more liquid in the morning compared with the rest of the day. The liquidity curves for 12:30 p.m. and 1:30 p.m. are very close and intersect at several points. Hence, it is virtually impossible to classify Alcatel's liquidity at these times.

4. The multi-asset framework

This section extends our analysis to a multivariate setup. We first develop joint activity measures for multiple assets and next extend them to accommodate trading intensities, volumes, and prices. This approach enhances the optimal portfolio allocation strategy with the potential benefit of exploiting the complementarity or substituability of assets. According to the ordering criterion introduced in the previous section, the investor is able to rank assets and retain the most efficient ones in his portfolio. Alternatively, the investor may wish to optimize portfolio composition in terms of the interactions among assets. He may essentially follow two strategies. The investor will pursue an aggressive strategy if he forms his portfolio from assets with positively correlated intensities. So an increase in the trading rate of one asset will trigger an increase in the trading rates of the other assets in the portfolio. In a more conservative scenario, the investor chooses assets with negatively correlated intensities, so that a decline in the frequency of trades for one asset can be compensated by increased trade intensity in the other assets.

This analysis becomes more complex when volume and prices are also considered. It can be viewed as a natural extension of the standard volatility – co-volatility matrix to an intensity–co-intensity framework with incorporated value and volume effects.

4.1. Activity–co-activity measures

Let us consider n assets indexed by $j, j = 1, \ldots, n$. The counting process measuring the numbers of trades for a particular asset j during a day m up to time t is denoted by $N_t^j(m)$. Using vector notation, we define the multivariate counting process: $N_t(m) = [N_t^1(m), \ldots, N_t^n(m)]'$, t varying. We assume that at most two trades of various assets may occur during an infinitesimal time interval and at most one per individual asset. The multi-asset intensity rates are defined by

$$\Pr\begin{bmatrix}\text{one trade between } t \text{ and } t + \mathrm{d}t\text{, day } m,\\ \text{and this trade is on asset } i\end{bmatrix} = \lambda_{ii}(t)\mathrm{d}t + \mathrm{o}(\mathrm{d}t), \tag{4.1}$$

$$\Pr\begin{bmatrix}\text{two trades between } t \text{ and } t + \mathrm{d}t\text{, day } m,\\ \text{and these trades are on assets } i \text{ and } j\end{bmatrix} = \lambda_{ij}(t)\mathrm{d}t + \mathrm{o}(\mathrm{d}t), \tag{4.2}$$

where: $\lambda_{ij}(t) = \lambda_{ji}(t)$, $i \neq j$,

$$\Pr\begin{bmatrix}\text{more than two trades between } t \text{ and } t + \mathrm{d}t,\\ \text{day } m\text{, on the same asset}\end{bmatrix} = \mathrm{o}(\mathrm{d}t), \tag{4.3}$$

and

$$\Pr\begin{bmatrix}\text{one trade on more than three}\\ \text{assets between } t \text{ and } t + \mathrm{d}t\text{, day } m\end{bmatrix} = \mathrm{o}(\mathrm{d}t). \tag{4.4}$$

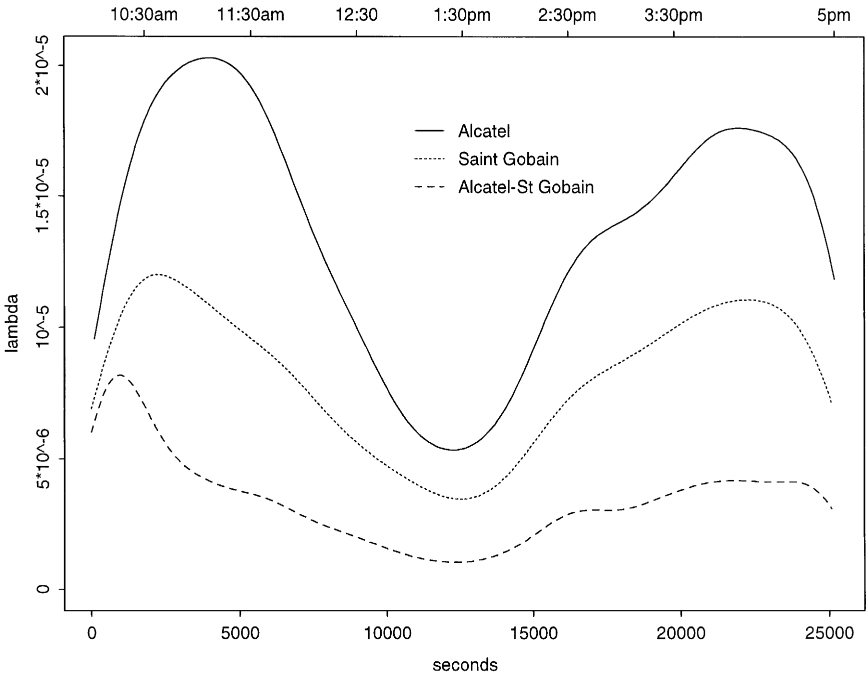

Fig. 15. Estimated individual and joint trading rates of Alcatel and Saint Gobain.

We denote by $\lambda_{ij}(t)$, $i \neq j$, the infinitesimal joint trading intensities of assets i and j, and by $\lambda_{ii}(t)$ the specific (idiosyncratic) trading rate of asset i at time t.

The marginal trade intensity rate for a single asset j introduced in Section 2.1 is equal to

$$\lambda_j(t) = \sum_{i=1}^{n} \lambda_{ij}(t). \tag{4.5}$$

The square matrix $A(t)$ with diagonal elements $\lambda_i(t)$ and off-diagonal elements $\lambda_{ij}(t)$, $i \neq j$, is the infinitesimal variance–covariance matrix of increments of the counting processes:

$$A(t)\mathrm{d}t \simeq V[\mathrm{d}N_t(m)],$$

where $\mathrm{d}N_t(m)$ stands for $N_{t+\mathrm{d}t}(m) - N_t(m)$. This is a symmetric nonnegative matrix, which may be used to compute the correlations between intensities:

$$\alpha_{ij}(t) = \frac{\lambda_{ij}(t)}{\lambda_i(t)^{1/2}\lambda_j(t)^{1/2}}. \tag{4.6}$$

As an illustration, we consider the case $n = 2$ and two specific stocks: Alcatel and Saint-Gobain. The joint trading rate is rather low compared to the marginal ones (see Fig. 15).

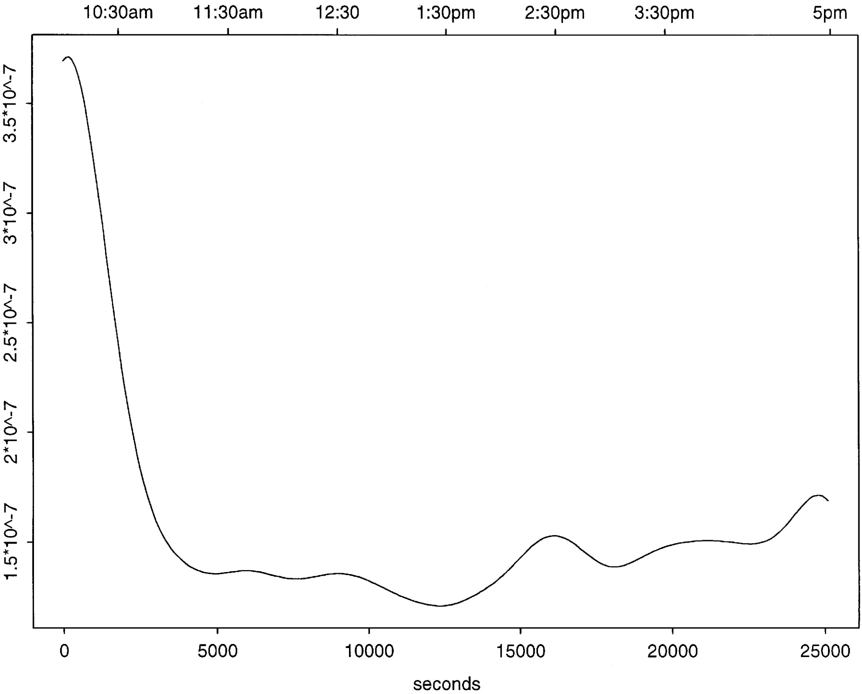

Fig. 16. Estimated correlation coefficient between Alcatel and Saint Gobain.

Indeed, the latter ones involve continuously substituting at least a hundred stocks regularly traded on the Paris Bourse for Alcatel. To plot the trading rates in the same figure, we multiply the joint rate by a factor 2×10^6. The three curves displayed in Fig. 15 have similar shapes. This resemblance may be due to the presence of a single underlying factor with different impacts on the rates. We verify this by plotting the estimated correlation coefficient $\alpha_{12}(t)$ between Alcatel and Saint-Gobain in Fig. 16. Since the activity correlations are significantly different from one, the one factor model is rejected. We observe that the activity correlation varies during the day and is always positive. The assets are rather strongly correlated after the market opening indicating the highest substitution rate between the two stocks at that time. The correlation decreases rapidly within the first trading hour and remains almost steady until noon.

4.2. Volume activity–co-activity measures

Let us now extend the co-activity measures by introducing volume. Consider the cumulated volumes corresponding to different assets:

$$V_t^j(m) = \sum_{n=1}^{N_t^j(m)} v_n^j(m), \tag{4.7}$$

and the associated multivariate process $V_t(m) = [V_t^1(m), \ldots, V_t^n(m)]'$. We define the volume activity–co-activity measures at time t by

$$A^v(t)\mathrm{d}t = V[\mathrm{d}V_t(m)]. \tag{4.8}$$

An increase in the aggregate traded volume may result from an increase in either the trading frequency, or the traded volume per transaction. It is important for a potential investor to distinguish these effects.

We introduce below the compound process $[V_t(m)]$ written as a stochastic integral with respect to the counting process $[N_t(m)]$. Let us also introduce a continuous-time latent volume process $v_t^*(m), t \in [0, 25\,200]$, such that $v_{d_n}^*(m) = v_n(m)$. We get

$$V_t^j(m) = \int_0^t v_u^*(m)\, \mathrm{d}N_m^j(u), \quad j = 1, \ldots, n, \tag{4.9}$$

or equivalently

$$V_t(m) = \int_0^t v_u^*(m)\, \mathrm{d}N_m(u). \tag{4.10}$$

Now let us introduce the following assumptions:

Assumption 2. The processes $[v_t^*(m)]$ and $[N_t(m)]$, m varying, are independent.

Assumption 3. The random vectors $v_t^*(m)$, t varying, are independent, with mean $\mu(t)$ and variance–covariance matrix $\Omega(t)$.

Under these assumptions, we derive a decomposition formula for the volume activity–co-activity (see Appendix D), whose components are easily interpreted.

$$\underbrace{a_{ij}^v(t)}_{\substack{\text{volume}\\ \text{co-activity}}} = \underbrace{a_{ij}(t)\mu_i(t)\mu_j(t)}_{\substack{\text{marginal effect of}\\ \text{frequency co-activity}}} + \underbrace{a_{ij}(t)w_{ij}(t)}_{\substack{\text{cross co-activity}\\ \text{frequency-volume}}} \tag{4.11}$$

This decomposition formula is essential for understanding the volume co-activity measure. Even if the two latent volumes per trade are uncorrelated, i.e. $w_{ij}(t) = 0$, the volume co-activity measure is not necessarily equal to zero if some correlation between the trading rates exists. The first term of the decomposition is always nonnegative, whereas the second one may take on any sign.

In Figs. 17 and 18 , we present estimates of the components of the decomposition formula. The cross activity term is positive indicating some substitutions between the assets and the peaks of the M-curve are not at the same level indicating a larger variability of the volume per trade in the morning than in the afternoon.

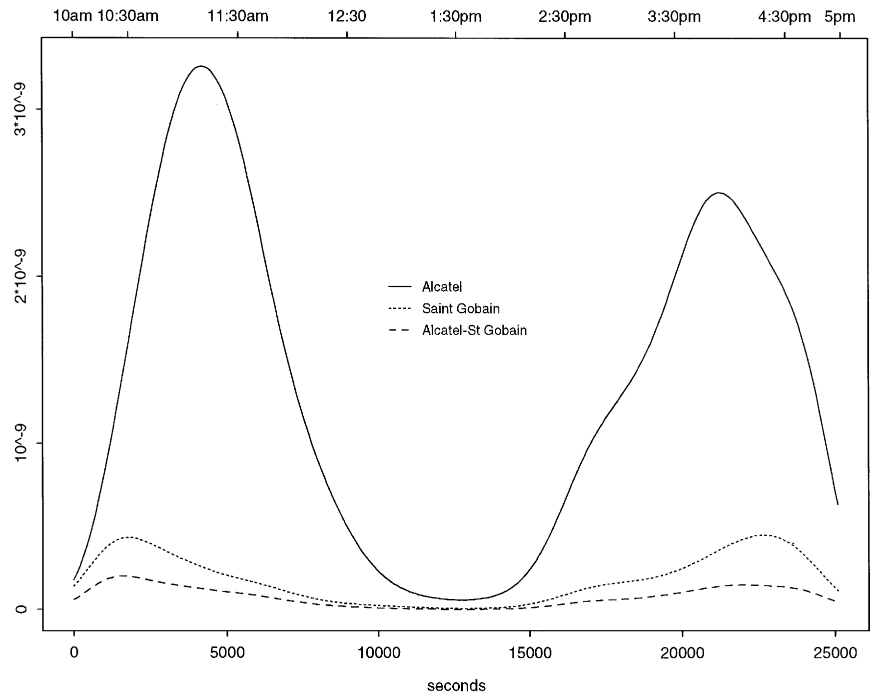

Fig. 17. Estimated marginal effect of transaction activity and co-activity Alcatel and Saint-Gobain.

5. Concluding remarks

This paper proposes empirical measures of market trading activity and liquidity for single assets and portfolios. The univariate measures highlight the importance of the appropriate timing of trading strategies, especially those involving large transactions. We also provide a liquidity-asset-based ordering and define liquidity costs and volatility. The multivariate measures allow us to study the interactions between assets. We show that the multivariate measures can assess the degree of substituability or complementarity of assets.

This analysis suggests a new framework for joint modeling of duration, volume and price processes as alternative to the ARCH specification (Lamoureux and Lastrapes, 1991; Engle and Russell, 1997,1998; Ghysels and Jasiak, 1996; Guillaume et al., 1995; Darolles et al., 1998). To accommodate the volume, time and price, three aggregated processes must be considered. A promising approach is to introduce the vector $[N_t(m), V_t(m), W_t(m)]$, where the first component accommodates trade intensity, the second one volume, and the third one trade value. Alternatively, we may also consider the trivariate process:

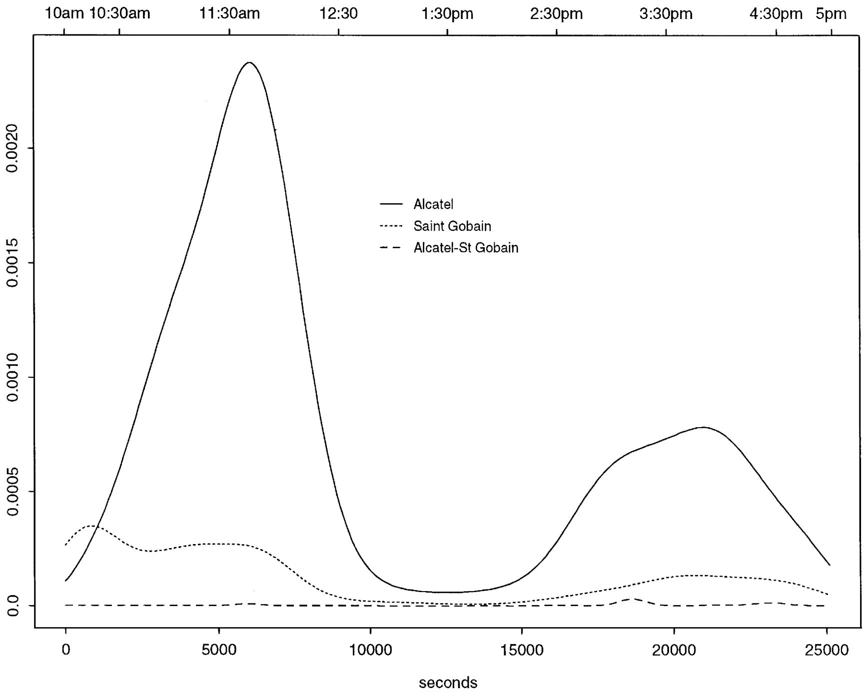

Fig. 18. Estimated cross activity transaction-volume Alcatel and Saint-Gobain.

$[N_t(m), V_t(m), p_t(m)]$, where $[p_t(m)]$ is the price process:

$$p_t(m) = \sum_{n=1}^{N_t(m)} \delta_n(m) = \int_0^t \delta_u^*(m)\, \mathrm{d}N_m(u),$$

where $\delta_n(m)$ denotes the price modification between successive ticks. Such an analysis would involve the variance–covariance matrix of $[\mathrm{d}N_t, \mathrm{d}V_t, \mathrm{d}p_t]$. The variance of $\mathrm{d}N_t$ and $\mathrm{d}V_t$ corresponds to the unweighted and volume-weighted activity–co-activity matrices. The variance of $\mathrm{d}p_t$ is the standard volatility–co-volatility matrix. Among the crossterms, the covariance between $\mathrm{d}V_t(m)$ and $\mathrm{d}p_t(m)$ is certainly worth investigating to fully describe the substitution effects.

Acknowledgements

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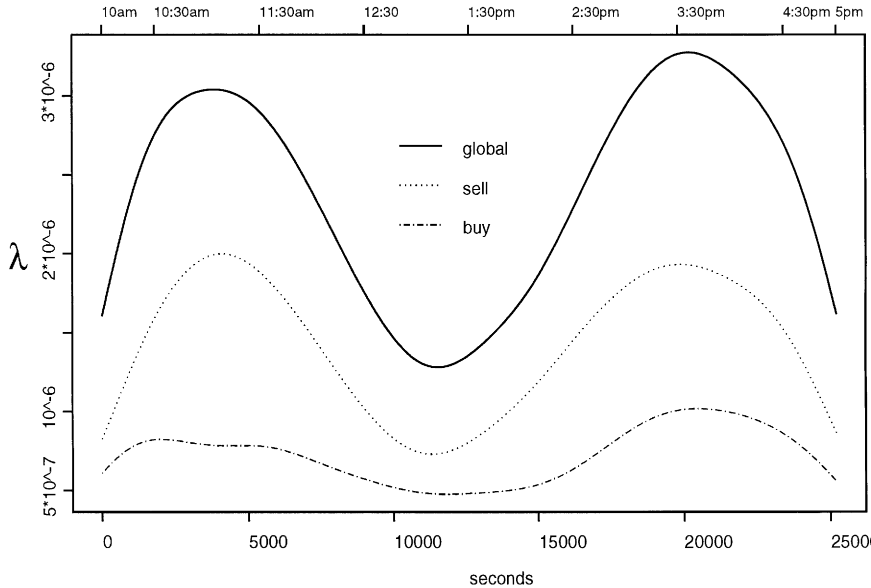

Fig. 19. Market order arrival rate, 1996, Alcatel.

Conference, San Diego, USA 1997, for their helpful comments. All errors are our own.

Appendix A. Market and limit order activity

We provide in the figures above and below the activity curves for the market and limit order arrivals (Figs. 19 and 20).

Appendix B

The data analysis is given in Table 1.

Data processing:

(1) From this table we see that for Alcatel, at 10:31:23 a.m. there was a trade called an "application" recorded by the electronic system. We drop all the "application" trades for two reasons. First of all, applications do not represent any liquidity, nor any possibility to trade for the other investors since an application is a sell/buy agreement, i.e. the investor arrive to the market with his/her counterpart. The second reason for eliminating these trades comes from the recorded time of the trade which is not always the time at which the trade

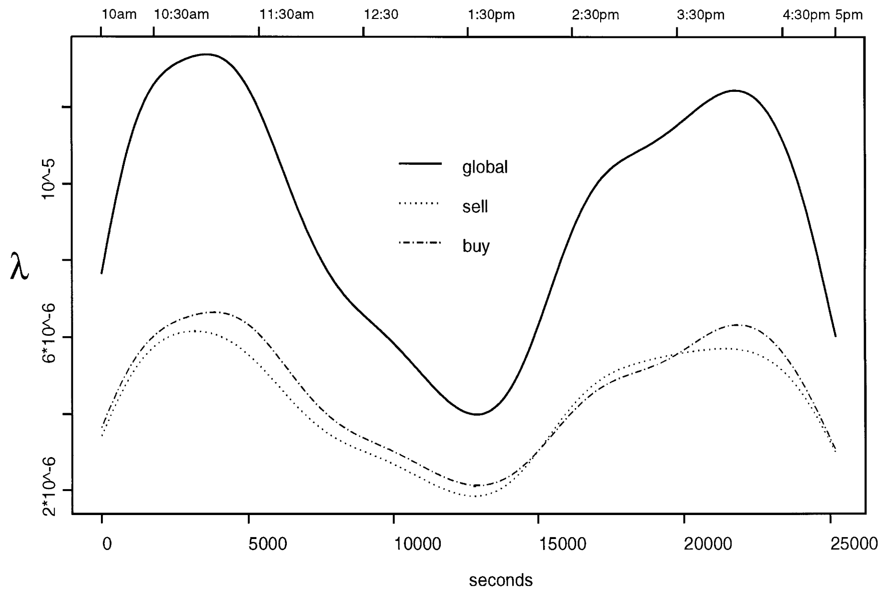

Fig. 20. Limit order arrival rate, 1996, Alcatel.

occurred, but the time at which the trade is introduced into the electronic system. Over the considered period, 169 "applications" were dropped.

(2) At 10:30:10 a.m. for the Alcatel stock, eight transactions are recorded giving at this precise time not only one market price, but six different market prices. This phenomenon is due to the stripping of a large order. Basically, all the trades recorded at 10:30:10 a.m. were initiated by the same investor. We then aggregate these trades, to end up with one market price which is the unitary price (458.57) the investor paid to receive the total volume (5000 shares):

$$\frac{458.3 \times 1060 + 458.4 \times 1050 + 458.5 \times 250 + 458.6 \times 1050 + 458.8 \times 1050 + 459 \times (350 + 90 + 100)}{5000}$$

$$= 458.57FF$$

Appendix C. Bid and ask curves as functions of volume

These curves are depicted in Fig. 21.

Appendix D. Decomposition of volume activity

We suppose that v^* and N are independent and

$$E[\mathrm{d}N_t] = \lambda(t)\mathrm{d}t, \quad V[\mathrm{d}N_t] = A(t)\mathrm{d}t$$

$$E[v_t^*] = \mu(t), \quad V[v_t^*] = \Omega(t).$$

Table 1
Table BDM, Alcatel-Alsthöm and Saint-Gobain

Internal code	Date	Time	Seq. NB	Price	Quantity	Appl. yes/no
Alcatel Alsthöm						
4438	3 Jan. 95	102945	1	458.2	490	0
4438	3 Jan. 95	102945	2	458.3	2510	0
4438	3 Jan. 95	103010	1	458.3	1060	0
4438	3 Jan. 95	103010	2	458.4	1050	0
4438	3 Jan. 95	103010	3	458.5	250	0
4438	3 Jan. 95	103010	4	458.6	1050	0
4438	3 Jan. 95	103010	5	458.8	1050	0
4438	3 Jan. 95	103010	6	459.0	350	0
4438	3 Jan. 95	103010	7	459.0	90	0
4438	3 Jan. 95	103010	8	459.0	100	0
4438	3 Jan. 95	103029	1	459.0	250	0
4438	3 Jan. 95	103029	2	459.0	2250	0
4438	3 Jan. 95	103029	3	459.0	500	0
4438	3 Jan. 95	103032	1	459.0	10	0
4438	3 Jan. 95	103051	1	457.6	20	0
4438	3 Jan. 95	103119	1	459.0	50	0
4438	3 Jan. 95	103123	1	458.5	460	1
Saint Gobain						
4322	3 Jan. 95	102601	1	615.0	10	0
4322	3 Jan. 95	103009	1	615.0	140	0
4322	3 Jan. 95	103150	1	615.0	30	0
4322	3 Jan. 95	103425	1	616.0	250	1
4322	3 Jan. 95	103709	1	615.0	10	0
4322	3 Jan. 95	103934	1	616.0	30	0
4322	3 Jan. 95	104000	1	616.0	10	0
4322	3 Jan. 95	104303	1	616.0	10	0
4322	3 Jan. 95	104320	1	617.0	100	0

Internal code : Stock's identification code from the data base (BDM).
Date : Transaction date.
Time : Transaction time (format: hhmmss).
Seq. NB : Sequence number, i.e. the ranking of orders if several transactions occurred at the same time.
Price : Transaction price.
Quantity : Transaction volume, i.e. the number of traded shares.
appl. yes/no : indicator which is equal to one if the transaction is a sell-buy "agreement" between two investors, which can be executed at a price beyond the bid–ask spread, zero otherwise (called an "application" on the Paris Bourse).

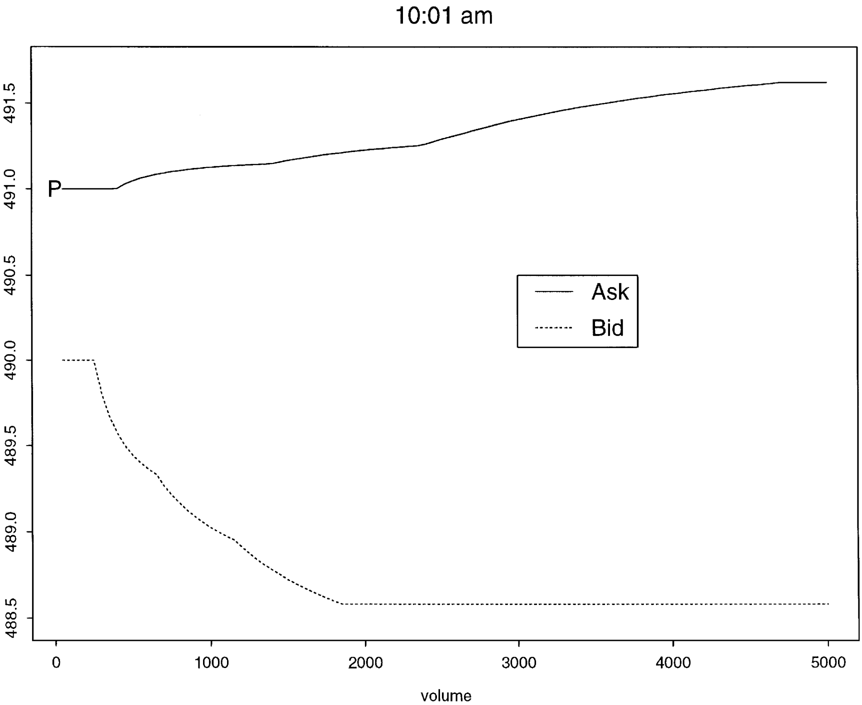

Fig. 21. Bid and ask curves and last market price (P) at 10:01 a.m., May the 14th, 1996, Alcatel.

Hence

$$E[\mathrm{d}V_t] = E[v_t^* \cdot \mathrm{d}N_t] = \mu(t) \cdot \lambda(t)\,\mathrm{d}t$$

The second order moment is such that

$$\begin{aligned} \mathsf{V}[\mathrm{d}V_t] &= \mathsf{V}[v_t^* \cdot \mathrm{d}N_t] \\ &= \mathsf{V}E[v_t^* \cdot \mathrm{d}N_t/N] + E\mathsf{V}[v_t^* \cdot \mathrm{d}N_t/N] \\ &= \mathsf{V}[\mu(t) \cdot \mathrm{d}N_t] + E[\operatorname{diag}(\mathrm{d}N_t) \cdot \Omega(t) \cdot \operatorname{diag}(\mathrm{d}N_t)] \\ &= \operatorname{diag}\mu(t) \cdot A(t) \cdot \operatorname{diag}\mu(t) + E[\operatorname{diag}(\mathrm{d}N_t) \cdot \Omega(t) \cdot \operatorname{diag}(\mathrm{d}N_t)], \end{aligned}$$

i.e. a matrix whose general term is given by

$$a_{ij}^v = a_{ij}(t)[\mu_i(t) \cdot \mu_j(t) + w_{ij}(t)].$$

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